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The spanwise perturbation of two-dimensional boundary layers

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Large spanwise variations of boundary-layer thickness and surface shear have been found recently in wind tunnels designed to maintain two-dimensional flow. Bradshaw (1965) argues that these variations are caused by minute deflexions in the free-stream flow rather than by any intrinsic instability of the boundary layers. This paper is a study of the effect of a small, periodic transverse flow on a flat-plate boundary layer. The perturbation flow Reynolds number is assumed to be $O(1)$ as it is in the experiments.

1. Introduction

In a series of wind-tunnel tests under nominally two-dimensional conditions, Klebanoff & Tidstrom (1959) found quasi-periodic spanwise variations of boundary-layer thickness of order $\pm 8\,\%$. Recently the phenomenon recurred in a National Physical Laboratory tunnel specifically designed for the study of two-dimensional boundary layers. Bradshaw (1965) sought a remedy as well as an explanation and found that these variations could result from lateral convergence or divergence of the flow downstream of slightly non-uniform settling-chamber damping screens. A rough analysis suggested that a boundary layer is surprisingly sensitive to spanwise velocity variations. The thickness variations found by Klebanoff could have been produced by variations in the free-stream flow direction of around 0·04 degree, much too small to be measured directly. This paper is a rigorous analysis of the effect of a small, periodic spanwise component of velocity on the boundary layer of a flat plate. The flow is assumed to be incompressible, steady and laminar.

Three-dimensional effects in the boundary layer will depend on the transverse flow field chosen for the incident flow. Suppose U_0 characterizes the chordwise component of free-stream flow, γU_0 the amplitude of the transverse perturbation. Suppose the frequency of the spanwise flow is specified by a wave-number k. The Reynolds number of the perturbation is then

$$R = \gamma U_0 / k\nu.$$

If Bradshaw's explanation is correct, the value of R corresponding to Klebanoff's data can be computed, and it is found to be around 3. It is not surprising that R is of order 1, since the transverse velocity variations are supposed to arise from the non-uniform drag of damping screens—a viscous phenomenon to begin with R will be regarded as a parameter of order 1 throughout the analysis.

2. Statement of the problem

The momentum and continuity equations are

$$\mathbf{U}.\nabla\mathbf{U} = -\frac{\nabla P}{\rho}+\nu\nabla^2\mathbf{U},$$
$$\nabla.\mathbf{U} = 0,$$

for a steady, incompressible flow field $\mathbf{U} = (U, V, W)$. The co-ordinates and physical situation are shown in figure 1. For a characteristic speed U_0 and

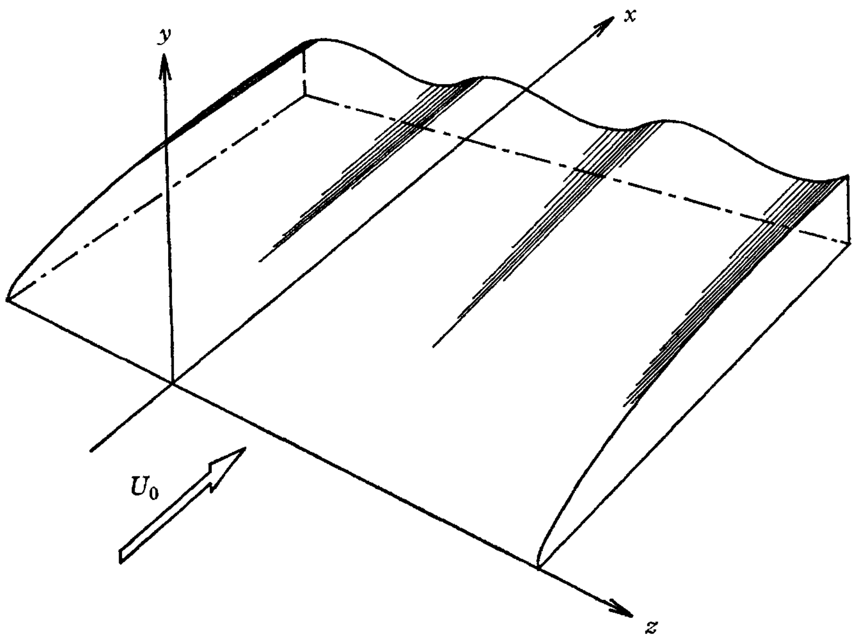

FIGURE 1. Sketch of co-ordinate system.

perturbation wave-number k, the following non-dimensional variables are appropriate:

$$(\xi, \eta, \zeta) = (kx, ky, kz),$$
$$(u, v, w) = \left(\frac{U}{U_0}, \frac{V}{U_0}, \frac{W}{U_0}\right),$$
$$p = \frac{P}{\rho U_0^2}.$$

The equations of motion in non-dimensional form are:

$$(\xi\text{-momentum})\quad uu_\xi+vu_\eta+wu_\zeta = -p_\xi+\epsilon^2(u_{\xi\xi}+u_{\eta\eta}+u_{\zeta\zeta}),$$
$$(\eta\text{-momentum})\quad uv_\xi+vv_\eta+wv_\zeta = -p_\eta+\epsilon^2(v_{\xi\xi}+v_{\eta\eta}+v_{\zeta\zeta}),$$
$$(\zeta\text{-momentum})\quad uw_\xi+vw_\eta+ww_\zeta = -p_\zeta+\epsilon^2(w_{\xi\xi}+w_{\eta\eta}+w_{\zeta\zeta}),$$
$$(\text{continuity})\quad u_\xi+v_\eta+w_\zeta = 0,$$

where $\epsilon^2 \equiv \nu k/U_0$. If γ is the amplitude of the angular variation of free-stream flow direction, the perturbation Reynolds number is

$$R = \gamma U_0/k\nu = \gamma/\epsilon^2 \sim O(1).$$

In Klebanoff's experiments γ was typically $0{\cdot}40° \sim 0{\cdot}001$ rad., so ϵ was about $0{\cdot}02$. In this paper ϵ is used as an expansion parameter in a perturbation scheme.

The boundary condition at the plate is

$$(u, v, w) = (0, 0, 0).$$

The upstream flow can be specified in any convenient way as long as the field chosen carries the desired transverse perturbation and is an adequate approximation to a solution of the equations of motion. Let the expansions for u and w in the outer flow begin

$$u = 1 + \ldots,$$
$$w = \gamma \cos \zeta + \ldots = R\epsilon^2 \cos \zeta + \ldots.$$

3. Solution far upstream

The velocity components above cannot be worked into a uniformly convergent solution to the equations of motion. Since the Reynolds number of the perturbation is of order 1, the transverse field of the incident stream must decay under the action of viscosity. Suppose we try a solution of the form

$$u = 1,$$
$$v = R(\xi)\, \epsilon^2 \eta \sin \zeta,$$
$$w = R(\xi)\, \epsilon^2 \cos \zeta,$$

where v has been chosen to satisfy the continuity equation. The approximate momentum equation

$$\mathbf{u}_\xi = \epsilon^2(\mathbf{u}_{\eta\eta} + \mathbf{u}_{\zeta\zeta})$$

is satisfied for

$$R(\xi) = R_0\, e^{-\epsilon^2 \xi}.$$

In fact, a uniformly convergent approximate solution to the equations of motion is

$$u = 1 + O(\epsilon^4),$$
$$v = R\epsilon^2 \eta \sin \zeta + O(\epsilon^4),$$
$$w = R\epsilon^2 \cos \zeta + O(\epsilon^4),$$
$$p = p_0 + \tfrac{1}{2} R^2 \epsilon^4 (\sin^2 \zeta - \eta^2) + O(\epsilon^6),$$

for that $R(\xi)$.

An expansion of the outer solution in powers of ϵ cannot be uniformly convergent. But such an expansion converges over an arbitrarily large interval $\Delta\xi$, where

$$\Delta\xi \ll 1/\epsilon^2.$$

As long as attention is confined to such an interval $\Delta\xi$, a straightforward expansion in powers of ϵ can be carried out, and the upstream boundary conditions may be taken as

$$u = 1 + O(\epsilon^4),$$
$$v = R\epsilon^2 \eta \sin \zeta + O(\epsilon^4),$$
$$w = R\epsilon^2 \cos \zeta + O(\epsilon^4),$$

where change in R is now contained in the $O(\epsilon^4)$ corrections.

4. Outer expansion

Let ξ, η, ζ remain fixed, and allow ϵ to tend to zero. The dependent variables are expanded in powers of ϵ as follows:

$$
\begin{aligned}
u &= 1+\epsilon f_1(\xi,\eta,\zeta)+\epsilon^2 f_2(\xi,\eta,\zeta)+\ldots,\\
v &= \epsilon g_1(\xi,\eta,\zeta)+\epsilon^2 g_2(\xi,\eta,\zeta)+\ldots,\\
w &= R\epsilon^2\cos\zeta+\ldots,\\
p &= p_0+\epsilon p_1(\xi,\eta,\zeta)+\epsilon^2 p_2(\xi,\eta,\zeta)+\ldots.
\end{aligned}
$$

When the coefficients of consecutive powers of ϵ in the equations of motion are set equal to zero, the following system of equations results:

ξ momentum

$$O(\epsilon) \quad f_{1\xi} = -p_{1\xi}, \tag{1}$$

$$O(\epsilon^2) \quad f_1 f_{1\xi}+f_{2\xi}+g_1 f_{1\eta} = -p_{2\xi}. \tag{2}$$

η momentum

$$O(\epsilon) \quad g_{1\xi} = -p_{1\eta}, \tag{3}$$

$$O(\epsilon^2) \quad f_1 g_{1\xi}+g_{2\xi}+g_1 g_{1\eta} = -p_{2\eta}. \tag{4}$$

ζ momentum

$$O(\epsilon) \quad p_{1\zeta} = 0, \tag{5}$$

$$O(\epsilon^2) \quad p_{2\zeta} = 0. \tag{6}$$

Continuity

$$O(\epsilon) \quad f_{1\xi}+g_{1\eta} = 0, \tag{7}$$

$$O(\epsilon^2) \quad f_{2\xi}+g_{2\eta}-R\sin\zeta = 0. \tag{8}$$

No boundary conditions are available at the plate. The outer expansion must be matched to an inner (boundary-layer) expansion there. In accordance with the discussion of the previous section, the conditions far upstream are $f_1, f_2, f_3, g_1 \to 0$. 'Far upstream' means $-\xi \gg 1$; we cannot really permit $-\xi\to\infty$, since the expansion form assumed is valid only in an interval $\Delta\xi \ll \epsilon^{-2}$.

5. Inner expansion

An expanded boundary-layer variable $\tilde{\eta} = \eta/\epsilon$ must be used to bring out the behaviour of the fluid near the plate. Then let ξ, $\tilde{\eta}$, ζ remain fixed, and allow ϵ to approach zero. The dependent variables are again expanded in powers of ϵ:

$$
\begin{aligned}
u &= F_0(\xi,\tilde{\eta},\zeta)+\epsilon F_1(\xi,\tilde{\eta},\zeta)+\epsilon^2 F_2(\xi,\tilde{\eta},\zeta)+\ldots,\\
v &= \epsilon G_1(\xi,\tilde{\eta},\zeta)+\epsilon^2 G_2(\xi,\tilde{\eta},\zeta)+\ldots,\\
w &= \epsilon^2 H_2(\xi,\tilde{\eta},\zeta)+\ldots,\\
p &= p_0+\epsilon P_1(\xi,\tilde{\eta},\zeta)+\epsilon^2 P_2(\xi,\tilde{\eta},\zeta)+\ldots.
\end{aligned}
$$

The equations of motion split up into the following system:

ξ momentum

$$O(1) \quad F_0 F_{0\xi} + G_1 F_{0\tilde{\eta}} = F_{0\tilde{\eta}\tilde{\eta}}, \tag{9}$$

$$O(\epsilon) \quad F_1 F_{0\xi} + F_0 F_{1\xi} + G_2 F_{0\tilde{\eta}} + G_1 F_{1\tilde{\eta}} = -P_{1\xi} + F_{1\tilde{\eta}\tilde{\eta}}, \tag{10}$$

$$O(\epsilon^2) \quad F_0 F_{2\xi} + F_1 F_{1\xi} + F_2 F_{0\xi} + G_1 F_{2\tilde{\eta}} + G_2 F_{1\tilde{\eta}} + G_3 F_{0\tilde{\eta}} + H_2 F_{0\zeta} = -P_{2\xi} + F_{0\xi\xi} + F_{2\tilde{\eta}\tilde{\eta}} + F_{0\zeta\zeta}. \tag{11}$$

η momentum

$$O(1) \quad P_{1\tilde{\eta}} = 0, \tag{12}$$

$$O(\epsilon) \quad F_0 G_{1\xi} + G_1 G_{1\tilde{\eta}} = -P_{2\tilde{\eta}} + G_{1\tilde{\eta}\tilde{\eta}}. \tag{13}$$

ζ momentum

$$O(\epsilon) \quad P_{1\zeta} = 0, \tag{14}$$

$$O(\epsilon^2) \quad F_0 H_{2\xi} + G_1 H_{2\tilde{\eta}} = -P_{2\zeta} + H_{2\tilde{\eta}\tilde{\eta}}. \tag{15}$$

Continuity

$$O(1) \quad F_{0\xi} + G_{1\tilde{\eta}} = 0, \tag{16}$$

$$O(\epsilon) \quad F_{1\xi} + G_{2\tilde{\eta}} = 0, \tag{17}$$

$$O(\epsilon^2) \quad F_{2\xi} + G_{3\tilde{\eta}} + H_{2\zeta} = 0. \tag{18}$$

At the plate all terms in the expansions of u, v, w are zero. Further conditions are provided by matching the inner and outer solutions in an intermediate region where they are simultaneously valid.

6. Matching

The forms assumed for the inner and outer expansions are valid only if the solutions based on them can be matched. Since matching must be done step-by-step in the analysis which follows, general equations for the procedure are derived here.

Consider the inner and outer expansions of any dependent variable a:

$$\text{(inner)} \quad a = A_0(\xi, \tilde{\eta}, \zeta) + \epsilon A_1(\xi, \tilde{\eta}, \zeta) + \epsilon^2 A_2(\xi, \tilde{\eta}, \zeta) + \dots,$$

$$\text{(outer)} \quad a = a_0(\xi, \eta, \zeta) + \epsilon a_1(\xi, \eta, \zeta) + \epsilon^2 a_2(\xi, \eta, \zeta) + \dots.$$

The matching is done on an intermediate variable $\eta^* = \eta/\lambda(\epsilon)$ such that, for η^* fixed and $\epsilon \to 0$,

$$\eta = \lambda(\epsilon)\eta^* \to 0, \quad \tilde{\eta} = \frac{\lambda(\epsilon)}{\epsilon}\eta^* \to \infty.$$

The outer solution may be expanded around $\eta = 0$ in the form

$$a = a_0 + \lambda_{\eta^*} a_{0\eta} + \epsilon a_1 + \epsilon\lambda\eta^* a_{1\eta} + \tfrac{1}{2}(\lambda\eta^*)^2 a_{0\eta\eta} + \epsilon^2 a_2 + \dots,$$

where the arguments of each function on the right are $(\xi, 0, \zeta)$. The inner solution has the form

$$a = A_0\left(\xi, \frac{\lambda}{\epsilon}\eta^*, \zeta\right) + \epsilon A_1\left(\xi, \frac{\lambda}{\epsilon}\eta^*, \zeta\right) + \epsilon^2 A_2\left(\xi, \frac{\lambda}{\epsilon}\eta^*, \zeta\right) + \dots.$$

In order for the two expansions to match for η^* fixed and $\epsilon \to 0$, the following conditions must hold:

$$A_0(\xi, \infty, \zeta) = a_0(\xi, 0, \zeta),$$
$$\lim_{\tilde{\eta}\to\infty} A_1(\xi, \tilde{\eta}, \zeta) = \tilde{\eta} a_{0\eta}(\xi, 0, \zeta) + a_1(\xi, 0, \zeta),$$
$$\lim_{\tilde{\eta}\to\infty} A_2(\xi, \tilde{\eta}, \zeta) = \tfrac{1}{2}\tilde{\eta}^2 a_{0\eta\eta}(\xi, 0, \zeta) + \tilde{\eta} a_{1\eta}(\xi, 0, \zeta) + a_2(\xi, 0, \zeta).$$

7. Initial steps in solving the problem

The solution must be carried to second order in ϵ to show the most interesting effects produced by the transverse field of the incident flow. The programme can be carried out by finding solutions to a sequence of groups of the equations (1)–(18). The functions F_0, F_1, G_1, G_2, H_2, f_1, f_2, g_1, g_2 are found that way in the five steps of this section. That is preliminary. The effect of the transverse field on the chordwise flow is uncovered only when F_2 is found, and that is deferred to § 8.

At the beginning of each of the steps below the ingredients needed are listed—the equations from the system (1)–(18), the boundary conditions, and the matching conditions.

First step—determining F_0 and G_1

equations:	(9), (16)	
boundary conditions:	$F_0(\xi, 0, \zeta) = 0,$	(*a*)
	$G_1(\xi, 0, \zeta) = 0,$	(*b*)
matching condition:	$F_0(\xi, \infty, \zeta) = 1.$	(*c*)

Let $F_0 = \psi_{\tilde{\eta}}$. Then equation (16) becomes

$$\psi_{\tilde{\eta}\xi} + G_{1\tilde{\eta}} = 0.$$

Hence
$$G_1 = -\psi_\xi + \text{fn}(\xi, \zeta),$$

where $\text{fn}(\xi, \zeta)$ is zero if (*b*) is satisfied by putting $\psi_\xi(\xi, 0, \zeta) = 0$. Equation (9) becomes

$$\psi_{\tilde{\eta}}\psi_{\tilde{\eta}\xi} - \psi_\xi\psi_{\tilde{\eta}\tilde{\eta}} = \psi_{\tilde{\eta}\tilde{\eta}\tilde{\eta}}.$$

Set
$$\psi = \sqrt{(2\xi)}\,\mathscr{F}(s), \quad s = \tilde{\eta}/\sqrt{(2\xi)}.$$

Then $\mathscr{F}(s)$ satisfies
$$\mathscr{F}''' + \mathscr{F}\mathscr{F}'' = 0.$$

F_0 and G_1 become
$$F_0 = \mathscr{F}'(s),$$
$$G_1 = (1/\sqrt{(2\xi)})\,[s\mathscr{F}'(s) - \mathscr{F}(s)],$$

so conditions (*a*), (*b*), (*c*) are

$$\mathscr{F}'(0) = \mathscr{F}(0) = 0, \quad \mathscr{F}'(\infty) = 1.$$

$\mathscr{F}(s)$ is thus the Blasius function. Suppose β is defined as follows:

$$\lim_{s\to\infty} \mathscr{F}(s) = s - \beta.$$

Then $$\lim_{\tilde{\eta}\to\infty} G_1(\xi,\tilde{\eta}) = (1/\sqrt{(2\xi)}) \lim_{s\to\infty} (s\mathscr{F}'-\mathscr{F}) = \beta/\sqrt{(2\xi)}.$$

Notice F_0 and G_1 do not depend on ζ.

Second step—determining H_2

equations: (5), (6), (13), (15)

boundary condition: $H_2(\xi, 0, \zeta) = 0,$ (*a*)

matching conditions: $H_2(\xi, \infty, \zeta) = R\cos\zeta,$ (*b*)

$$\lim_{\tilde{\eta}\to\infty} P_2(\xi,\tilde{\eta},\zeta) = \tilde{\eta}p_{1\eta}(\xi,0,\zeta)+p_2(\xi,0,\zeta). \quad (c)$$

The matching conditions here, as elsewhere, are applications of the general matching equations derived earlier. (*c*), for example, is the second-order matching condition for p with $p_{0\eta\eta} = 0$. Equation (13) may be written

$$P_{2\tilde{\eta}} = G_{1\tilde{\eta}\tilde{\eta}} - F_0 G_{1\xi} - G_1 G_{1\tilde{\eta}}.$$

Since F_0 and G_1 do not depend on ζ, $P_{2\tilde{\eta}\zeta} = 0$, so $P_{2\zeta} = \text{fn}(\xi,\zeta)$. Differentiating (*c*) on ζ and using equations (5) and (6) yield $\lim_{\tilde{\eta}\to\infty} P_{2\zeta} = 0$. Thus

$$P_{2\zeta} = 0$$

everywhere. Equation (15) then becomes

$$F_0 H_{2\xi} + G_1 H_{2\tilde{\eta}} = H_{2\tilde{\eta}\tilde{\eta}},$$

which is the same as equation (9) if $H_2 = \text{fn}(\zeta) F_0(\xi,\tilde{\eta})$. The solution satisfying conditions (*a*) and (*b*) is

$$H_2 = RF_0\cos\zeta = R\mathscr{F}'(s)\cos\zeta.$$

Thus the spanwise flow follows the Blasius profile to the order considered.

Third step—determining f_1, g_1 and p_1

equations: (1), (3), (5), (7)

boundary conditions: $f_1, g_1 \to 0$ far upstream, (*a*)

matching condition: $G_1(\xi,\infty) = g_1(\xi,0,\zeta).$ (*b*)

Equations (1) and (3), $f_{1\xi} = -p_{1\xi}$ and $g_{1\xi} = -p_{1\eta}$, combine to give the equation for conservation of spanwise vorticity,

$$(g_{1\xi} - f_{1\eta})_\xi = 0.$$

By the upstream conditions (*a*), $$g_{1\xi} - f_{1\eta} = 0.$$

Equations (1), (3) and (5) then imply

$$p_1 = -f_1.$$

Since the spanwise vorticity is zero, there is a potential function ϕ such that

$$f_1 = \phi_\xi, \quad g_1 = \phi_\eta,$$

and equation (7) becomes $$\phi_{\xi\xi} + \phi_{\eta\eta} = 0.$$

Condition (*b*) and the expression for $G_1(\xi, \infty)$ found in the first step give

$$\phi_\eta(\xi, 0) = \beta/\surd(2\xi)$$

over the plate. ϕ is thus the linearized potential for flow around a thin parabolic cylinder (van Dyke 1964). The solution satisfies

$$f_1(\xi, 0) = -p_1(\xi, 0) = 0$$

next to the plate, and f_1 and g_1 do not depend on ζ.

Fourth step—determining F_1 and G_2

equations: (10), (12), (14), (17)

boundary conditions: $F_1(\xi, 0, \zeta) = 0,$ (*a*)

$G_2(\xi, 0, \zeta) = 0,$ (*b*)

matching conditions: $F_1(\xi, \infty, \zeta) = f_1(\xi, 0) = 0,$ (*c*)

$P_1(\xi, \infty, \zeta) = p_1(\xi, 0) = 0.$ (*d*)

Since equations (12) and (14) imply $P_1 = P_1(\xi)$, condition (*d*) requires $P_1 = 0$. Equations (10) and (17) are thus homogeneous and linear in F_1 and G_2, and the only solution compatible with conditions (*a*), (*b*) and (*c*) is

$$F_1 = G_2 = 0.$$

Fifth step—determining f_2, g_2 and p_2

equations: (2), (4), (6), (7), (8)

boundary condition: $f_2 \to 0$ far upstream, (*a*)

matching condition: $\tilde{\eta} g_{1_\eta}(\xi, 0) + g_2(\xi, 0, \zeta) = \lim_{\tilde{\eta} \to \infty} G_2 = 0.$ (*b*)

By equation (7) $g_{1_\eta} = -f_{1_\xi}$, and from the third step $f_1(\xi, 0) = 0$. Hence

$$g_{1_\eta}(\xi, 0) = 0,$$

and (*b*) becomes

$$g_2(\xi, 0, \zeta) = 0.$$

In the third step it was shown that $f_{1_\eta} = g_{1_\xi}$, so equations (2), (4), (6) and (8) can be written

$$\begin{aligned} f_{2_\xi} &= -\{\tfrac{1}{2}(f_1^2 + g_1^2) + p_2\}_\xi, \\ g_{2_\xi} &= -\{\tfrac{1}{2}(f_1^2 + g_1^2) + p_2\}_\eta, \\ p_{2_\xi} &= 0, \\ f_{2_\xi} + g_{2_\eta} &= R \sin \zeta, \end{aligned}$$

and the solution satisfying conditions (*a*) and (*b*) is

$$\begin{aligned} f_2 &= 0, \\ g_2 &= R_\eta \sin \zeta, \\ p_2 &= -\tfrac{1}{2}(f_1^2 + g_1^2). \end{aligned}$$

8. Final steps to determine F_2

In the last section it was shown that the first-order correction to the chordwise boundary-layer profile is zero. If the theory is going to account for the large boundary-layer thickness and shear variations observed by Klebanoff and Bradshaw, those effects will have to show up in the function F_2 yet to be calculated. The trouble is that even in the strictly two-dimensional case there is a second-order correction to the Blasius profile. Since the perturbation equations are linear in the functions still uncomputed, solutions can be superposed, and the contribution of the transverse field can be separated from the two-dimensional part of the solution. The two-dimensional part decreases toward zero downstream, but the part driven by the transverse field increases rapidly.

The pressure function P_2

By means of the expressions for F_0 and G_1 derived in the last section, equation (13) can be written

$$P_{2\tilde{\eta}} = \frac{1}{2\sqrt{(2)}\,\xi^{\frac{3}{2}}}[\mathscr{F}'' + s\mathscr{F}'^2 - \mathscr{F}\mathscr{F}'].$$

$P_{2\zeta}$ was found to be zero. If P_2 takes the form

$$P_2 = \mathscr{P}(s)/\xi,$$

then $\mathscr{P}$ must satisfy

$$\mathscr{P}' = \tfrac{1}{2}[\mathscr{F}'' + s\mathscr{F}'^2 - \mathscr{F}\mathscr{F}'].$$

As $s \to \infty$, $\mathscr{P}' \to \frac{1}{2}\beta$, and the form assumed for P_2 is valid only if that limit is compatible with the matching condition

$$\lim_{\tilde{\eta}\to\infty} P_2 = \tilde{\eta}p_{1\eta}(\xi, 0) + p_2(\xi, 0).$$

But from equation (3) and the work of § 7

$$p_{1\eta}(\xi, 0) = -g_{1\xi}(\xi, 0) = -G_{1\xi}(\xi, \infty) = \beta/2\xi\sqrt{(2\xi)},$$

$$p_2(\xi, 0) = -\tfrac{1}{2}g_1^2(\xi, 0) = -\beta^2/4\xi.$$

Hence

$$\lim_{\tilde{\eta}\to\infty} P_2 = \beta s/2\xi - \beta^2/4\xi,$$

which is compatible with the form assumed earlier if the constant of integration for $\mathscr{P}$ is chosen such that

$$\mathscr{P}(s) \to \tfrac{1}{2}\beta s - \tfrac{1}{4}\beta^2 \quad \text{as} \quad s \to \infty.$$

Transformation of the equation for F_2

Since $F_1 = G_2 = F_{0\zeta} = 0$ and $H_2 = RF_0 \cos\zeta$, equations (11) and (18) are

$$F_0F_{2\xi} + F_2F_{0\xi} + G_1F_{2\tilde{\eta}} + G_3F_{0\tilde{\eta}} = -P_{2\xi} + F_{0\xi\xi} + F_{2\tilde{\eta}\tilde{\eta}},$$

$$F_{2\xi} + G_{3\tilde{\eta}} = R\sin\zeta\, F_0.$$

Let $F_0 = \psi_{\tilde{\eta}}$ as before, and let $F_2 = \chi_{\tilde{\eta}}$. Then the continuity equation becomes

$$\chi_{\xi\tilde{\eta}} + G_{3\tilde{\eta}} = R\sin\zeta\, \psi_{\tilde{\eta}}.$$

Hence

$$G_3 = R\sin\zeta\psi - \chi_{\xi} + \text{fn}\,(\xi, \zeta)$$

and fn $(\xi, \zeta) = 0$ if the boundary condition $G_3(\xi, 0, \zeta) = 0$ is satisfied by requiring $\chi_\xi(\xi, 0, \zeta) = 0$. Now transform

$$(\xi, \tilde{\eta}, \zeta) \Rightarrow (\xi, s, \zeta), \quad s = \tilde{\eta}/\sqrt{(2\xi)},$$

in the momentum equation. Thus

$$\chi(\xi, \tilde{\eta}, \zeta) = X(\xi, s, \zeta),$$

and F_0, G_1 and P_2 are already known in terms of the new variables. The equation becomes

$$X_{sss} + \mathscr{F} X_{ss} - 2\xi\, \mathscr{F}' X_{s\xi} + 2\xi \mathscr{F}'' X_\xi + \mathscr{F}' X_s$$
$$= 2\sqrt{(2)}\, \xi^{\frac{3}{2}} R \sin\zeta\, \mathscr{F}\mathscr{F}'' - \frac{2\sqrt{2}}{\sqrt{\xi}} [\mathscr{P} + \tfrac{1}{2}s\mathscr{P}' + \tfrac{1}{4}(s^2\mathscr{F}''' + 3s\mathscr{F}'')].$$

The boundary conditions at the plate are

$$X_s(\xi, 0, \zeta) = 0, \quad X_\xi(\xi, 0, \zeta) = 0,$$

and, say, $$\chi(\xi, 0, \zeta) = 0.$$

The matching condition for F_2 is

$$\lim_{\tilde{\eta} \to \infty} F_2 = \tilde{\eta} f_{1\eta}(\xi, 0) + f_2(\xi, 0).$$

From the last section $f_2 = 0$ and $f_{1\eta}(\xi, 0) = g_{1\xi}(\xi, 0) = -\beta/2\xi\sqrt{(2\xi)}$.

Hence $$\lim_{\tilde{\eta} \to \infty} F_2 = -\beta s/2\xi,$$

and $$\lim_{s \to \infty} X_s(\xi, s, \zeta) = -\beta s/\sqrt{(2\xi)}.$$

It is easy to show by direct substitution that that limit is compatible with the transformed momentum equation.

Separation of X into two- and three-dimensional parts

In the transformed momentum equation there is one term which is modulated by $R \sin\zeta$; there are no such terms in the boundary conditions. That term reflects the $R \sin\zeta\psi$ part of G_3 and is a forcing function imposed by the transverse field through the continuity condition. X can be written as a sum of two parts, one proportional to $R \sin\zeta$ and the other not involving ζ at all. The first term responds to the forcing function proportional to $R\sin\zeta$ and obeys zero boundary conditions all around. The second term responds to the two-dimensional forcing function and satisfies the X_s limit for $s \to \infty$. Thus write

$$X = \frac{1}{\sqrt{(2\xi)}} [R\xi^2 \sin\zeta\, \mathscr{J}(s) + \mathscr{I}(s)].$$

$\mathscr{J}(s)$ and $\mathscr{I}(s)$ are defined by separate differential equations and boundary conditions:

$$\left.\begin{aligned} \mathscr{J}''' + \mathscr{F}\mathscr{J}'' - 2\mathscr{F}'\mathscr{J}' + 3\mathscr{F}''\mathscr{J} &= 4\mathscr{F}\mathscr{F}'', \\ \mathscr{J}(0) = \mathscr{J}'(0) = 0, \quad \mathscr{J}'(\infty) &= 0, \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathscr{I}''' + \mathscr{F}\mathscr{I}'' + 2\mathscr{F}'\mathscr{I}' - \mathscr{F}''\mathscr{I} &= -4\mathscr{P} - 2s\mathscr{P}' - s^2\mathscr{F}''' - 3s\mathscr{F}'', \\ \mathscr{I}(0) = \mathscr{I}'(0) = 0, \quad \lim_{s \to \infty} \mathscr{I}(s) &= -\beta s. \end{aligned}\right\}$$

If the spanwise vorticity is to decay exponentially far from the plate, $\mathscr{I}(s)$ must contain an $O(\log \epsilon)$ term (Van Dyke 1964). There is no need to find out more about $\mathscr{I}(s)$. The important point is that the two-dimensional contribution X approaches zero as ξ becomes large, and the three-dimensional term grows as $\xi^{\frac{3}{2}}$.

F_2 and the boundary-layer profile

The $\mathscr{J}$ equation with its boundary conditions has a simple solution—

$$\mathscr{J} = \mathscr{F} - s\mathscr{F}',$$

that can be verified by direct substitution using the Blasius equation

$$\mathscr{F}''' + \mathscr{F}\mathscr{F}'' = 0$$

and its derivative. Then

$$F_2 = \chi_{\xi} = X_s/\sqrt{(2\xi)} = -\tfrac{1}{2}R\xi \sin \zeta s\mathscr{F}'' + \mathscr{I}'/2\xi.$$

The boundary-layer profile is

$$u = \mathscr{F}'(s) - \tfrac{1}{2}R\epsilon^2\xi \sin \zeta s\mathscr{F}''(s) + \epsilon^2(\mathscr{I}'(s)/2\xi) + O(\epsilon^3).$$

Notice the expansion is not uniformly convergent. The second term is much smaller than the first only if $\xi \ll 1/\epsilon^2$, but that is assured by the restriction $\Delta\xi \ll 1/\epsilon^2$ already imposed to make the outer flow tractable. The third term is small if $\xi \gg \epsilon^2$, the usual requirement for convergence of the boundary-layer expansion.

The first two terms of the profile expansion can be combined into a single function

$$\mathscr{F}'\left(\frac{s}{1+\frac{1}{2}R\epsilon^2\xi \sin \zeta}\right)$$

with third-order accuracy. Then

$$u = \mathscr{F}'(s^*) + (\epsilon^2/2\xi)\,\mathscr{I}'(s^*) + O(\epsilon^2),$$

where

$$s^* = s/(1+\tfrac{1}{2}\gamma\xi \sin \zeta),$$

and $\gamma = R\epsilon^2$. Thus the shape of the profile is unaffected by the transverse field. Even in the second-order approximation, the only three-dimensional effect is a spanwise variation in boundary-layer thickness.

9. Conclusion

For the profile expansion to be valid, ξ must satisfy $\epsilon^2 \ll \xi \ll 1/\epsilon^2$. In physical variables the inequality can be written

$$\nu k/U_0 \ll kx \ll R/\gamma,$$

and in that interval, expressions good to $O(\gamma)$ for U and W are

$$U = U_0[\mathscr{F}'(y/\delta) + \nu/U_0 x \mathscr{I}'(y/\delta)],$$
$$W = \gamma U_0 \cos kz\, \mathscr{F}'(y/\delta),$$

where

$$\delta = \frac{2\nu x}{U_0}(1+\tfrac{1}{2}\gamma kx \sin kz).$$

Thus the boundary layer takes on the wavy character illustrated in figure 1. The practical significance of these results is discussed by Bradshaw (1965).

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